Star Formation

Q & A Session 30.06.2020

Protostellar Disks & Outflows

Magnetic Torque in a thin disk

> Derive an estimate of the torque acting on a geometrically thin accretion disc due to a dipolar magnetic field that originates in a rotating central star. Assume that the disc has an inner edge located away from the star at radius R_{min}.

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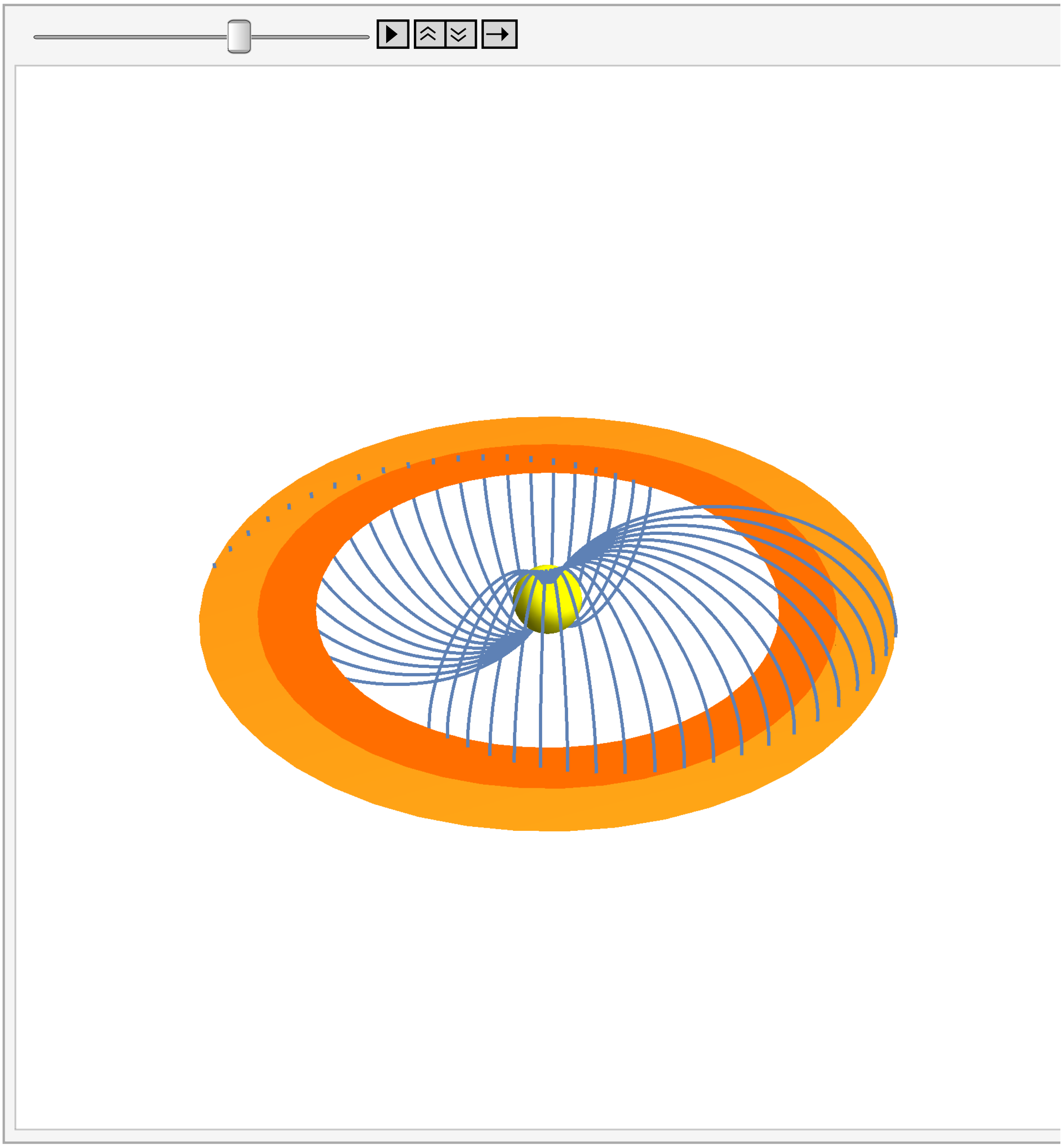


The estimate of the total torque exerted on the inner part of the accretion disc by the magnetic field is

$$\mathcal{T} = \frac{4\pi}{3} \frac{B_z^2(R_*)}{\mu_0} \frac{R_*^6}{R_{min}^3} \tag{1}$$

where $B_z(R_*)$ is the component of the magnetic flux density in the direction perpendicular to the disc at the surface of the star, R_* is the radius of the star, and μ_0 is the permeability of free space.

By equating this magnetic torque to the internal viscous torque acting at the inner radius R_{min}, derive an expression for the inner radius of an accretion disc that is truncated by the stellar magnetic field.

The inner radius R_{min} and the radius of the central star R_* are related by

$$\frac{R_{min}}{R_*} = \left(\frac{4\,\pi\,B_z^2\,(R_*)\,R_*^{5/2}}{3\,\sqrt{G\,M}\,\dot{m}\,\mu_0}\right)^{2/7} \tag{2}$$

where $\dot{m}$ is the mass flow rate through the disc, and M is the mass of the central star. G is the constant of gravitation.

Consider a 1 $M_\odot$ T Tauri star of radius $R_* = 1\,R_\odot$, with a magnetic field strength at its surface of $B_z(R_*) = 10^{-1}$ Tesla. If the star is accreting at a rate $\dot{M} = 10^{-8}\,M_\odot\,\mathrm{yr}^{-1}$, calculate the radius of the inner edge of the accretion disc.

$$\frac{R_{min}}{R_*} = 50\left(\frac{B_z\,(R_*)}{\mathrm{Tesla}}\right)^{4/7}\left(\frac{R_*}{R_\odot}\right)^{5/7}\left(\frac{\dot{m}}{10^{-8}\,M_\odot\,\mathrm{yr}^{-1}}\right)^{-2/7}\left(\frac{M}{M_\odot}\right)^{-1/7} \tag{3}$$

For the T Tauri star, putting $M = 1\,M_\odot$, $R_* = 1\,R_\odot$, $B_z(R_*) = 10^{-1}$ Tesla, $\dot{m} = 10^{-8}\,M_\odot\,\mathrm{yr}^{-1}$, we get an inner radius of the accretion disc

$$\frac{\mathbf{R_{min}}}{\mathbf{R_*}} = \mathbf{50\,(10^{-1})^{4/7}\,(1)^{5/7}\,(1)^{-2/7}\,(1)^{-1/7} = 13.413478976398627`}$$

$= 13.4\,R_\odot$

One class of young stars, known as FU Orionis stars, are known to undergo outbursts in which the apparent accretion rate increases substantially above the canonical value of $\dot{M} = 10^{-8}\,M_\odot\,\mathrm{yr}^{-1}$. If the accretion rate during outburst increases by a factor of 10^4 above this value, then calculate the radius of the inner edge of the accretion disc using the above stellar parameters. What do you think happens to the magnetic field in this case ?

$$\frac{\mathbf{R_{min}}}{\mathbf{R_*}} = \mathbf{50\,(10^{-1})^{4/7}\,(1)^{5/7}\,(10^4)^{-2/7}\,(1)^{-1/7} = 0.9653488644416252`}$$

This figure is actually smaller than the radius($R_* = 1\,R_\odot$) used for the star itself.

In this case the magnetic field lines are swept in towards the star by the increased mass flow and are essentially crushed against the stellar surface. The configuration is now very different since a boundary layer is generated.

A Simple T Tauri Disk Model

In this problem we will construct a simple model of a T Tauri star disk in terms of a few parameters: the midplane density and temperature ρ_m and T_m, the surface temperature T_s, the angular velocity Ω, and the specific opacity of the disk material κ. We assume that the disk is very geometrically thin and optically thick, and that it is in thermal and mechanical equilibrium.

a)

Assume that the disk radiates as a blackbody at temperature T_s. Show that the surface and midplane temperatures are related approximately by

$$T_m \approx \left(\frac{3}{8}\,\kappa\,\Sigma\right)^{1/4}\,T_s$$

where Σ is the disk surface density

The disk interior is optically thick, so the vertical radiation flux F is given by the diffusion approximation:

$$F = \frac{c}{3\kappa\rho}\frac{d}{dz}E = \frac{c\,a}{3\kappa\rho}\frac{d}{dz}\left(T^4\right) = \frac{4\sigma}{3\kappa\rho}\frac{d}{dz}\left(T^4\right)$$

where E is the radiation energy density and T is the gas temperature. In thermal equilibrium the flux does not vary with z, so we can re-arrange this equation and integrate from the midplane at $z = 0$ to the surface at $z = z_s$:

$$F\int_0^{z_s}\rho\, dz = \frac{4\sigma}{3\kappa}\int_{T_m}^{T_s}\frac{d}{dz}\left(T^4\right) dz$$

$$F\frac{\Sigma}{2} = \frac{4\sigma}{3\kappa}\left(T_m^4 - T_s^4\right)$$

$$F \approx \frac{8\sigma}{3\kappa\Sigma}T_m^4$$

where the factor of 2 in the denominator on the LHS in the second step comes from the fact that Σ is the column density of the entire disk, and we integrated over only half of it. In the third step we assumed that $T_m^4 >> T_s^4$, which will be true for any optically thick disk. Note that this is the flux carried away from the disk midplane in both the $+z$ and $-z$ directions – formally the flux changes direction discontinuously at $z = 0$ in this simple model, so the total flux leaving the midplane is twice this value.

If the disk radiates as a blackbody, the radiation flux per unit area leaving each side of the disk surface is σT_s^4 , and this must balance the flux that is transported upward through the disk by diffusion. Thus we have

$$\frac{8\sigma}{3\kappa\Sigma}T_m^4 \approx \sigma T_s^4$$

where the expressions on either side of the equality represent the fluxes in either the $+z$ or $-z$ directions either; the total fluxes are a factor of 2 greater, but the factors of 2 obviously cancel. Solving for T_m gives the desired result:

$$T_m \approx \left(\frac{3}{8}\kappa\Sigma\right)^{1/4} T_s$$

b)

Suppose the disk is characterized by a standard α model, meaning that the viscosity $\nu = \alpha c_s H$, where H is the scale height and c_s is the sound speed. For such a disk the rate per unit area of the disk surface (counting each side separately) at which energy is released by viscous dissipation is $F_d = (9/8)\,\nu\Sigma\Omega^2$. Derive an estimate for the midplane temperature T_m in terms of Σ, Ω, and α.

Equating the dissipation rate F_d per unit area with the radiation rate per unit area σT^4

$$\sigma T_s^4 = \frac{9}{8}\nu\Sigma\Omega^2$$

$$T_s = \left(\frac{9}{8}\frac{\nu\Sigma\Omega^2}{\sigma}\right)^{1/4} = \left(\frac{9}{8}\alpha\frac{c_s^2\Sigma\Omega}{\sigma}\right)^{1/4}$$

In turn, plugging this into the relation we just derived between the surface and midplane temperatures gives

$$T_m \approx \left(\frac{27}{64} \frac{\nu \kappa \Sigma^2 \Omega^2}{\sigma}\right)^{1/4} \approx \left(\frac{27}{64} \frac{\alpha \kappa c_s^2 \Sigma^2 \Omega}{\sigma}\right)^{1/4}$$

Substituting $c_s^2 = k_B T_m / \mu$, where μ is the mean particle mass and solving for T_m gives

$$T_m \approx \left(\frac{27}{64} \frac{\alpha \kappa k_B \Sigma^2 \Omega}{\sigma \mu}\right)^{1/4}$$

Note that it makes much more sense to compute c_s from the midplane temperature than from the surface temperature, since the vast majority of the viscous dissipation is occurring near the mid-plane, not at the disk surface.

c)

Calculate the cooling time of the disk in terms of the orbital period. Should the behavior of the disk be closer to isothermal or adiabatic?

The cooling time is the thermal energy divided by the energy radiation rate. The thermal energy per unit area is

$$E_{th} \approx \frac{\Sigma c_s^2}{\gamma - 1} = \frac{k_B \Sigma T_m}{(\gamma - 1) \mu}$$

where γ is the ratio of specific heats for the gas, which for molecular hydrogen will be somewhere between 5/3 and 7/5 depending on the gas temperature. The radiation rate is $2 \sigma T_s^4$, so the cooling time is

$$t_{cool} = \frac{E_{th}}{2 \sigma T_s^4} \approx \frac{k_B \Sigma T_m}{2 (\gamma - 1) \mu \sigma T_s^4} \approx \frac{3 \kappa k_B \Sigma^2}{16 (\gamma - 1) \mu \sigma T_m^3} \approx \frac{4}{9 (\gamma - 1) \alpha \Omega}$$

The orbital period is $t_{orb} = 2 \pi / \Omega$, so the ratio of cooling time to orbital period is

$$\frac{t_{cool}}{t_{orb}} \approx \frac{2}{9 \pi (\gamma - 1) \alpha}$$

For the typical values of α expected due to MRI or similar mechanisms, ~0.01 or less, this number is significantly bigger than unity, so the cooling time is longer than the orbital period. Under these conditions the disk is likely to act adiabatically rather than isothermally. Only if a gets quite large, ~0.1 or more, do we approach the isothermal regime

d)

Consider a disk with a mass of 0.03 $M_\odot$ orbiting a 1 $M_\odot$ star, which has $\kappa = 3\,\mathrm{cm}^2\, g^{-1}$ and $\alpha = 0.01$. The disk runs from 1 to 20 AU, and the surface density varies as R^{-1}. Use your model to express ρ_m, T_m, and T_s as functions of the radius, normalized to 1 AU; i.e., derive results of the form $\rho_m = r_0 (r/\mathrm{AU})^p$ for each of the quantities listed. Is your numerical model disk gravitationally unstable (i.e., $Q < 1$) anywhere?

Let the disk surface density be $\Sigma = \Sigma_0 (r/r_0)^{-1}$, and let $r_0 =$ 1 AU and $r_1 =$ 20 AU be the inner and outer radii. The mass in the disk is

$$M_{disk} = \int_{r_0}^{r_1} \Sigma_0 \left(\frac{r}{r_0}\right)^{-1} 2 \pi r \, dr = 2 \pi \Sigma_0 r_0 (r_1 - r_0)$$

so

$$\Sigma = \frac{M_{disk}}{2\pi r_0 (r_1 - r_0)} \left(\frac{r}{r_0}\right)^{-1} = 2.2 \times 10^3 \left(\frac{r}{1\ \text{AU}}\right)^{-1} \text{g cm}^{-2}$$

For a 1 $M_\odot$ star, the angular velocity of the orbit is

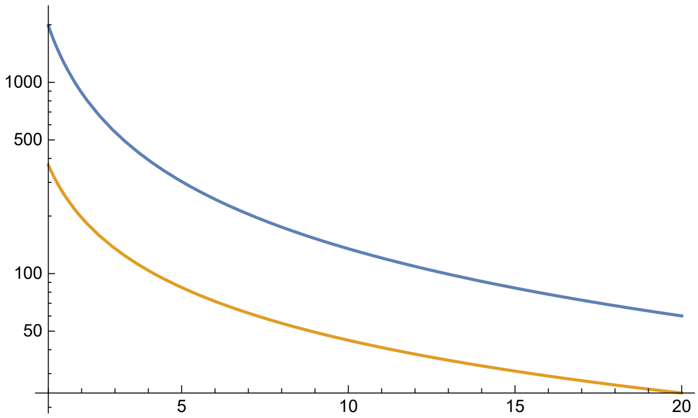
$$\Omega = \sqrt{\frac{G M}{r^3}} = 2.0 \times 10^{-7} \left(\frac{r}{1\ \text{AU}}\right)^{-3/2} \text{s}^{-1}$$

Plugging in $\kappa = 3\,\text{cm}^{-2}\, g^{-1} 1$ and $\alpha = 0.01$, taking $\mu = 3.9 \times 10^{-24}$ as the mean particle mass, and plugging into the expression for T_m derived in part (b) gives

$$T_m \approx 1980 \left(\frac{r}{1\ \text{AU}}\right)^{-7/6} \text{K}$$

In[76]:= $\textbf{LogPlot}[\{1980\, r^{-7/6}, 370\, r^{-11/12}\}, \{r, 1, 20\}]$

Out[76]=



and plugging this into the relation between T_m and T_s derived in part (a) gives

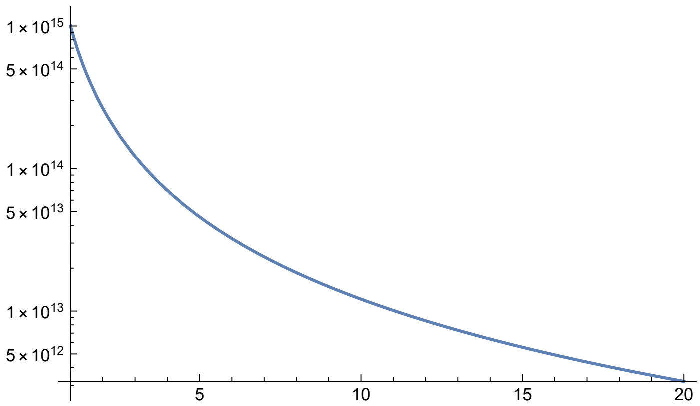
$$T_s \approx 370 \left(\frac{r}{1\ \text{AU}}\right)^{-11/12} \text{K}$$

The midplane density is $\rho_m \approx \Sigma / H$, where H is the scale height is $H = c_s / \Omega = \Omega^{-1} \sqrt{k_B T / \mu}$. If we use $T \approx T_m$ to compute the scale height, then we have

$$\rho_m \approx \frac{\Sigma\, \Omega}{\sqrt{k_B T_m / \mu}} = 1.7 \times 10^{-9} \left(\frac{r}{1\ \text{AU}}\right)^{-23/12} \text{g cm}^{-3}$$

In[78]:= $\textbf{LogPlot}\left[\left\{\frac{1.7 \times 10^{-9}}{1.7 \times 10^{-24}} r^{-23/12}\right\}, \{r, 1, 20\}\right]$

Out[78]=



Finally, the Toomre Q of the disk computed using the midplane temperature (which is the most reasonable one to use, since it is the temperature of most of the mass) is

$$Q = \frac{\Omega\, c_s}{\pi\, G\, \Sigma} = \frac{\Omega \sqrt{k_B\, T_m / \mu}}{\pi\, G\, \Sigma} = 110 \left(\frac{R}{1\ \mathrm{AU}}\right)^{-13/12}$$

This reaches a minimum value of 4.4 at *r* = 20 AU. Thus the disk is gravitationally stable.